Micro-Note on solving super p=2 form A_{MN}

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1 Introduction

Superforms are extended versions of differential forms which include superspace coordinates. For example a one-form superform can be written as,

$$A = dz^M A_M = dx^m A_m + d\theta^\alpha A_\alpha + d\bar{\theta}_{\dot{\alpha}} \bar{A}^{\dot{\alpha}} \tag{1}$$

The purpose of this note is to solve the constraints and Bianchi identities for a super 2-form Gauge field. An analogous calculation has been done in [1] for 1-form gauge field. We will proceed similarly. Variation of a p-form gauge field is given in terms of a (p-1)-form.

$$\delta A_p = dK_{p-1} \tag{2}$$

Field strength of a gauge p-form is given as,

$$F_{p+1} = dA_p \tag{3}$$

Bianchi identities are given as,

$$dF_{p+1} = 0 \tag{4}$$

All the details of "d" operator in superspace are given in [1].

2 Antisymmetric Gauge Field

Second rank antisymmetric gauge field are contained in chiral spinor superfield. Chiral spinor superfield is given as [2],

$$V_\alpha = \eta_\alpha + \theta_\alpha(A(x) + iB(x)) + \theta^\beta F_{\alpha\beta} + \theta^2(\chi_\alpha + i\partial_{\alpha\dot{\beta}}\dot{\eta}^{\dot{\beta}}) \tag{5}$$

Its gauge transformation is given as,

$$\delta\Psi_\alpha = i\bar{D}^2 D_\alpha K \tag{6}$$

The spinor superfield strength is given as,

$$G = D^\alpha \Psi_\alpha + \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} \tag{7}$$

So from this we know that once we enter the superfield language every super gauge field component can be represented in terms of chiral spinor superfield.

3 Field Strengths

Here we have listed field strengths of p=1,2,3 and 4 superforms. Following from (2-4) equations for p-form become Bianchi identities for (p-1)-form and hence those equations must be set to 0. Similarly variation of p-forms are given by (p-1)-form set of equations.

3.1 p=1

Field strength of the zero-form A is a one-form,

$$F_\alpha = D_\alpha A \tag{8}$$

$$F_a = \partial_a A \tag{9}$$

3.2 p=2

$$F_{ab} = \partial_a A_b - \partial_b A_a \tag{10}$$

$$F_{\alpha\beta} = D_{(\alpha} A_{\beta)} \tag{11}$$

$$F_{\alpha b} = D_\alpha A_b - \partial_b A_\alpha \tag{12}$$

$$F_{\alpha\dot{\beta}} = D_\alpha \bar{A}_{\dot{\beta}} + \bar{D}_{\dot{\beta}} A_\alpha - i2\sigma^c_{\alpha\dot{\beta}} A \tag{13}$$

The gauge transformation of 1-form gauge field can be read from last section,

$$\delta A_\alpha = D_\alpha B \tag{14}$$

$$\delta A_a = \partial_a B \tag{15}$$

here B is just a zero-form.

3.3 p=3

$$F_{\alpha\beta\gamma} = D_{(\alpha} A_{\beta\gamma)} \tag{16}$$

$$F_{\alpha\beta\dot{\gamma}} = D_{(\alpha} A_{\beta)\dot{\gamma}} + \bar{D}_{\dot{\gamma}} A_{\alpha\beta} + i2\sigma^c_{(\alpha\dot{\gamma}} A_{\beta)c} \tag{17}$$

$$F_{\alpha\beta c} = D_{(\alpha} A_{\beta)c} + \partial_c A_{\alpha\beta} \tag{18}$$

$$F_{\alpha\dot{\beta}c} = D_\alpha A_{\dot{\beta}c} + \bar{D}_{\dot{\beta}} A_{\alpha c} + \partial_c A_{\alpha\dot{\beta}} + i\sigma^d_{\alpha\dot{\beta}} A_{cd} \tag{19}$$

$$F_{\alpha bc} = D_\alpha A_{bc} - \partial_b A_{\alpha c} - \partial_c A_{\alpha b} \tag{20}$$

$$F_{abc} = \partial_a A_{bc} - \partial_b A_{ca} - \partial_c A_{ab} \tag{21}$$

3.4 p=4

$$B_{\alpha\beta\nu\delta} = D_{(\alpha}F_{\beta\nu\delta)} \tag{22}$$

$$B_{\alpha\beta\nu\dot{\delta}} = D_{(\alpha}F_{\beta\nu)\dot{\delta}} + \bar{D}_{\dot{\delta}}F_{\alpha\beta\nu} - i2\sigma^{d}_{(\alpha|\dot{\delta}}F_{|\beta\nu)d} \tag{23}$$

$$B_{\alpha\beta\dot{\nu}\dot{\delta}} = D_{(\alpha}\bar{F}_{\beta)\dot{\nu}\dot{\delta}} + \bar{D}_{(\dot{\nu}}F_{\alpha\beta\dot{\delta})} - i2\sigma^{d}_{(\alpha(\dot{\nu}|}F_{|\beta)\dot{\delta})d} \tag{24}$$

$$B_{\alpha\beta\nu d} = D_{(\alpha}F_{\beta\nu)d} - \partial_{d}F_{\alpha\beta\nu} \tag{25}$$

$$B_{\alpha\beta\dot{\nu}d} = D_{(\alpha}F_{\beta)\dot{\nu}d} + \bar{D}_{\dot{\nu}}F_{\alpha\beta d} - \partial_{d}F_{\alpha\beta\dot{\nu}} - i2\sigma^{d}_{(\alpha\dot{\nu}}F_{\beta)de} \tag{26}$$

$$B_{\alpha\beta cd} = D_{(\alpha}F_{\beta)cd} + \partial_{[c}F_{\alpha\beta d]} \tag{27}$$

$$B_{\alpha\dot{\beta}cd} = D_{\alpha}F_{\dot{\beta}cd} - \bar{D}_{\dot{\beta}}F_{\alpha cd} + \partial_{[c}F_{\alpha\dot{\beta}d]} - i\sigma^{e}_{\alpha\dot{\beta}}F_{cde} \tag{28}$$

$$B_{\alpha bcd} = D_{\alpha}F_{bcd} - \partial_{[b}F_{\alpha cd]} \tag{29}$$

$$B_{abcd} = \partial_{[a}F_{bcd]} \tag{30}$$

These equations become the Bianchi identities for a 2-form gauge field and hence each of these should vanish.

4 Constraints

To consistently solve Bianchi identities (22-30) for a 2-form we need certain constraints on the Field strengths. The reason for constraints is that when we went from an ordinary form to a superform we increased the degrees of freedom. To restore the actual number we need to impose certain constraints (for eg putting certain field strengths to 0) such that we get our original dofs back. The constraints should be consistent with Bianchi identities.

To begin with we will impose the following constraints,

$$F_{\alpha\beta\gamma} = 0 \qquad F_{\alpha\beta\dot{\gamma}} = 0 \qquad F_{\alpha\beta c=0} \tag{31}$$

Let us first look at $F_{\alpha\beta\gamma} = 0$ constraint. This trivially satisfies the Bianchi identity (22). Now to solve it we look at field strength equation (16)

$$D_{\alpha}A_{\beta\gamma} + D_{\beta}A_{\gamma\alpha} + D_{\gamma}A_{\alpha\beta} = 0 \tag{32}$$

Introducing a new spinor superfield U_{β} we see that the following form of $A_{\alpha\beta}$ solves the constraint,

$$A_{\alpha\beta} = D_{\alpha}U_{\beta} + D_{\beta}U_{\alpha} \tag{33}$$

Thus we have solved $A_{\alpha\beta}$ in terms of a spinor superfield U_{β}. (31b) and (31c) again trivially satisfy the Bianchi identity [23]. Now plugging the constraints in [24] we get,

$$F_{\alpha\dot{\beta}c} = -i\sigma_{c\alpha\dot{\beta}}G \tag{34}$$

Here G is a Superfield. Thus overall we have the constraint (31) and (34). So now we have everything we need to solve for A_{MN}. These constraints are sufficient to solve the 2-form.

5 Solution

We will solve for A_{MN} using equations(16-21) and (31,34). In the last section we have already found $A_{\alpha\beta}$. Using constraint (31b) in field equation (17) we get $A_{\beta c}$ in terms of $A_{\beta\dot{\nu}}$ and U_α,

$$i2\sigma^c_{(\alpha\dot{\nu}}A_{\beta)c} = D_{(\alpha}A_{\beta)\dot{\nu}} + \frac{1}{2}\bar{D}_{\dot{\nu}}A_{\alpha\beta} \tag{35}$$

Now we plug this in the field strength equation (18) and also use constraint (31c). Finally we get the solution as,

$$A_{\alpha\dot{\beta}} = D_\alpha\bar{U}_\beta + \bar{D}_{\dot{\beta}}U_\alpha - i2\sigma^c_{\alpha\dot{\beta}}U_c \tag{36}$$

The U_c superfield was added as it would vanish in a symmetric combination. Now we will solve for $A_{\alpha b}$ Using constraint (31) in Field equation (18) we get,

$$D_\alpha A_{\beta\dot{\nu}} + D_\beta A_{\alpha\dot{\nu}} + \bar{D}_{\dot{\nu}}A_{\alpha\beta} + i2\sigma^c_{(\alpha\dot{\nu}}A_{\beta)c} = 0 \tag{37}$$

We have put equation (36) to use here,

$$D_\alpha\bar{D}_{\dot{\nu}}U_\beta + D_\beta\bar{D}_{\dot{\nu}}U_\alpha + \bar{D}_{\dot{\nu}}D_\alpha U_\beta + \bar{D}_{\dot{\nu}}D_\beta U_\alpha - 2i\sigma^d_{(\alpha\dot{\nu}}D_{\beta)}U_d + i2\sigma^c_{(\alpha\dot{\nu}}A_{\beta)c} = 0 \tag{38}$$

Simplifying this we get,

$$-i2\sigma^c_{(\alpha\dot{\nu}}A_{\beta)c} = 2i\sigma^d_{(\alpha\dot{\nu}}D_{\beta)}U_d + i2\sigma^c_{(\alpha\dot{\nu}}A_{\beta)c} \tag{39}$$

From this we can read of the expression $A_{\beta c}$.

$$A_{\beta c} = D_\alpha U_b - \partial_b U_\alpha + i\sigma_{b\alpha\dot{\nu}}\bar{V}^{\dot{\nu}} \tag{40}$$

The additional chiral superfield V^α was added since it does not appear in a symmetric combination of $\sigma^c_{(\alpha\dot{\nu}}A_{\beta)c}$.

In a similar manner A_{ab} can be found. To summarize we have found the following results for A_{MN},

$$A_{\alpha\beta} = D_\alpha U_\beta + D_\beta U_\alpha \tag{41}$$

$$A_{\alpha\dot{\beta}} = D_\alpha\bar{U}_{\dot{\beta}} + \dot{D}_{\dot{\beta}}U_\alpha - i2\sigma^c_{\alpha\dot{\beta}}U_c \tag{42}$$

$$A_{\alpha b} = i\sigma^b_{\alpha\dot{\nu}}\bar{T}^{\dot{\nu}} + D_\alpha U_b - \partial_b U_\alpha \tag{43}$$

$$A_{ab} = \frac{i}{4}[\sigma^{\nu\delta}_{ab}D_\nu T_\delta + \bar{\sigma}^{\dot{\nu}\dot{\delta}}_{ab}\bar{D}_{\dot{\nu}}\bar{T}_{\dot{\delta}}] + \partial_a U_b - \partial_b U_a \tag{44}$$

$$D_\alpha\bar{V}_\beta = 0 \tag{45}$$

6 Transformation

With this we have expressed all components of A_{MN} in terms of U_a,U_α and V_α. Now we know that variation of a p-form is given in terms of (p-1)-form from equation (11). So we need to have,

$$\delta A_{\alpha\beta} = D_\alpha B_\beta + D_\beta B_\alpha \tag{46}$$

Now from (33) we have,

$$\delta A_{\alpha\beta} = D_\alpha \delta U_\beta + D_\beta \delta U_\alpha \tag{47}$$

Thus we can infer from (46) and (47) that,

$$\delta U_\alpha = B_\alpha + iD_\alpha \Lambda \tag{48}$$

Here Λ is a real superfield. Following a similar procedure we find that,

$$\delta T_\alpha = i\bar{D}^2 D_\alpha \Lambda \tag{49}$$

$$\delta U_\alpha = B_\alpha + iD_\alpha \Lambda \tag{50}$$

$$\delta U_a = B_a + \sigma_a^{\beta\dot{\nu}} [D_\beta, \bar{D}_{\dot{\nu}}]\Lambda \tag{51}$$

B_a and B_α can be used to gauge away U_α and U_a. Thus everything an be written in terms of spinor superfield V_α. From the Field Strength equation (19) we find the form of G,

$$G = \frac{1}{2}(D^\alpha V_\alpha + \bar{D}^{\dot{\alpha}} \bar{V}_{\dot{\alpha}}) \tag{52}$$

This matches with the expected value and thus, we have fully solved the equations.

7 Conclusion

We saw that to solve Bianchi identities we need the following constraints.

$$F_{\alpha\beta\gamma} = 0 \qquad F_{\alpha\beta\dot{\gamma}} = 0 \qquad F_{\alpha\beta c=0} \tag{53}$$

$$F_{\alpha\dot{\beta}c} = -i\sigma_{c\alpha\dot{\beta}} G \tag{54}$$

This gave us each component of A_{MN} in terms of a spinor superfield V_α, just like we wanted as in [2]. We also saw that the field strength was given as G (52) whose form again matched with [2]. Hence we consistently solved the p=2 superform.

The non-trivial aspect of this calculation is finding the constraints. In this short note our main concern was to solve the two form relations and hence not a lot of thought was put in the origin of these constraints. For a detailed account on how to find the constraints refer to [4].

8 References

1. J.Wess and J.Bagger,“Supersymmetry and Supergravity”,Princeton Univ Press

2. Warren Siegel,“Gauge Spinor Supefields as Scalar Multiplet”, Physics Letter Vol 85B,number 4

3. S. James Gates,“Super p-form Gauge Superfields”, Nucl. Phys. B184 (1981) 381-390

4. W.D. Linch III and S. Randall,“Superspace de Rham Complex and Relative Cohomology”, JHEP 09(2015)190 ,[hep-th/1412.4686]